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Effect of Piezoelectric Layer on Beam Parameters using Zigzag Theory

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***Abstract*— An efficient higher order theory is presented for static analysis of multilayered composite beams with piezoelectric layers embedded or bonded to the surface, under static electromechanical load. In this theory, the in-plane displacement field is taken as a combination of a layer-wise linear variation and a cubic variation across the thickness. Transverse normal strains are neglected. The electric field is also approximated as piecewise linear across the sub layers. The displacement field is expressed in terms of only three primary displacement variables excluding electric potential variables by enforcing the conditions of zero transverse shear stress at top and bottom of the beam and its continuity at layer interfaces under general electromechanical loading. The effect of thickness of the piezo-layer is observed for various loading conditions. Also, the effect of beam lay-up on various parameters is studied.**

***Index Terms*—zigzag theory, hybrid beam, piezoelectric layer, static analysis, thickness, electromechanical loading.**

I. Introduction

Hybrid beams and plates are made of an elastic substrate with piezoelectric layers embedded or bonded to its surface. Due to inhomogeneity in mechanical properties across the thickness and presence of piezo-layers, analysis of these structures requires appropriate electromechanical modeling. 3D analytical solutions have been presented for the piezoelectric response for simply supported rectangular plates in Heyliger, 1994 [1]; Lee and Jiang, 1996 [2]. Early works on 2D theories for hybrid laminated beams were based on classic laminate theory approximation for mechanical field without considering the coupling between mechanical and electric fields (Tzou, 1989 [3]). The limitation of CLT not including shear deformation effect was addressed by First Order Shear Deformation Theory (FSDT) (Kapuria et al., 1997b [4]) and refined third order theory of Reddy (1984) [5].

This paper presents an efficient higher order zigzag theory for the static analysis of hybrid piezoelectric beams subjected to electromechanical loads. The displacement field is taken as a combination of a linear variation layerwise and a cubic variation across the thickness. Transverse normal strains are neglected. The displacement field is expressed in terms of only three primary displacement variables excluding electric potential variables by enforcing the conditions of zero transverse shear stress at top and bottom of the beam and its continuity at layer interfaces under general electromechanical loading. The governing equations and boundary conditions are derived from the principle of virtual work and analytical Navier solution is obtained using Fourier series expansion. The effect of thickness of piezoelectric layer is examined for various beams keeping the total thickness of beam same in all cases and varying the thickness of piezo-layer. The results are plotted for various beam span-to-thickness ratios. The effect of beam lay-up on various beam parameters is also studied and the results are given for cross-ply and angle-ply beams.

II. Formulation Of Theory

A. Geometry of the Hybrid Beam

Consider a hybrid beam of solid cross-section of width b and thickness h, consisting of orthotropic layers with their principal material axis along the fibers at an arbitrary angle to the Cartesian coordinate axis-x along its length. The geometry of the beam is shown in Fig. 1. Some of the layers can be piezoelectric with class mm2 symmetry and poling along the thickness axis-z. The origin of the coordinate system (x,z) is taken at the middle surface of the beam. The integer k denotes the layer number which starts from the bottom of the laminate. The distance from the reference plane to the bottom surface of the k^{th} layer is denoted by z_{k-1}. Each layer is perfectly bonded to its adjacent plies.

B. Constitutive Equations

For a piezoelectric continuum which exhibits class mm2 symmetry with respect to principal material axes x_1, x_2, $x_3(=z)$

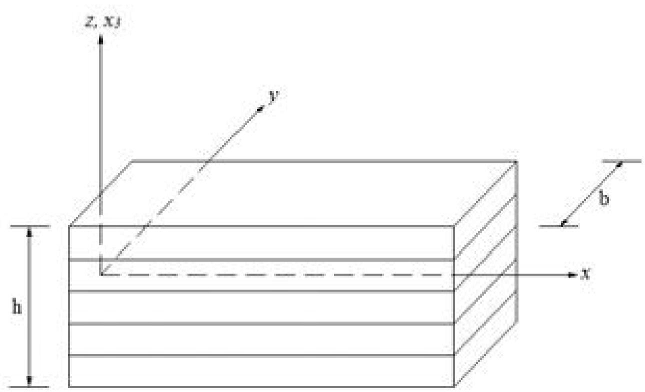

Figure 1. Geometry of the hybrid beam

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and is polarized along direction z, the constitutive equations referring to the orthogonal coordinate system (x,y,z) are given by (Auld, 1973) [6]; Tzou and Bao, 1995 [7] as

$$\varepsilon = \bar{S}\sigma + \bar{d}^T E, \quad D = \bar{d}\sigma + \bar{\epsilon} E \tag{1}$$

where the superscript T denotes matrix transpose. $\bar{S}$, $\bar{d}$, and $\bar{\epsilon}$ are respectively the transformed compliance coefficients, piezoelectric strain constants and constant stress dielectric constants.

For a beam with small width, the following assumptions are made:

$$\sigma_z, \sigma_y, \tau_{yz}, \tau_{xy}, E_y \cong 0 \tag{2}$$

On use of (2), the constitutive relations in (1) reduce to

$$\begin{bmatrix} \sigma_x \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & 0 \\ 0 & \bar{Q}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \gamma_{zx} \end{bmatrix} - \begin{bmatrix} 0 & \bar{e}_{31} \\ \bar{e}_{15} & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_z \end{bmatrix}$$

$$\begin{bmatrix} D_x \\ D_z \end{bmatrix} = \begin{bmatrix} 0 & \bar{e}_{15} \\ \bar{e}_{31} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \gamma_{zx} \end{bmatrix} + \begin{bmatrix} \bar{\eta}_{11} & 0 \\ 0 & \bar{\eta}_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_z \end{bmatrix} \tag{3}$$

C. Displacements and Potential Field

The presented 1D beam model combines a third order zigzag approximation for the in plane displacement u through the thickness of the laminate with a layerwise approximation for the electric potential ϕ such that the continuity of the transverse shear stress at the interfaces between adjacent layers is satisfied in presence of the electromechanical field. The transverse displacement field w is obtained by superimposing a constant field across the thickness with a layerwise contribution that accounts for the piezoelectric out-of-plane normal strain induced by the electric potential. Accordingly, the displacement and the potential fields are assumed in the following form:

$$u(x,z) = u_k(x) - zw_0(x)_{,x} + z\psi_k(x)$$

$$+z^2\xi(x) + z^3\eta(x) + \bar{a}^k \sum_{j=1}^{N} f^j(z)\phi^j_{,x} \tag{4a}$$

$$w(x,z) = w_0(x) - \bar{d}^k_{33} \sum_{j=1}^{N} \Psi^j(z)\phi^j$$

$$\phi(x,z) = \sum_{j=1}^{N} \Psi^j(z)\phi^j(x) \tag{4b}$$

with $\bar{a}^k = \bar{d}^k_{33} - \bar{e}^k_{15} / \bar{Q}^k_{55}$ and $f^j(z) = \int_0^z \psi^j(z)dz$. Here a subscript comma denotes differentiation. u_k and Ψ_k denote displacement and rotation variables of the kth layer. N is the number of points z^j across the laminate thickness for describing the variation of the potential field in the thickness direction. ϕ^j are the electric potentials at points z^j and $\Psi^j(z)$ are linear interpolation functions.

The functions u_k, Ψ_k, ξ and η are determined using the following conditions:

1. The top and bottom surfaces of the beam are traction-free i.e. $\tau_{zx}|_{z=\pm h/2} = 0$.
2. The shear stress is continuous across each layer interface.
3. The displacement u is continuous across each layer interface.

The final expression of u is obtained as:

$$u = u_0 - zw_{0,x} + R_k(z)\psi_1 + \sum_{j=1}^{N} F_k^j(z)\phi^j_{,x} \tag{5}$$

where

$$R_k(z) = R_1^k + zR_2^k + z^2R_3 + z^3R_4, \quad F_k^j(z) = \bar{a}^k f^j(z) + R_o^{kj}$$

It can be seen that above displacement field contains three primary variables (u_0, w_0, Ψ_1) excluding the contribution from the potential variables ϕ^j. The number of the primary variables is thus the same as in the FSDT.

D. Governing Equations

The equilibrium equations and the variationally consistent boundary conditions can be formulated in a weak form using the principle of virtual work given by:

$$\int_V (\sigma_i \delta\varepsilon_i + D_i \delta\phi_{,i})dV = \int_A (p_i \delta u_i - q\delta\phi)dA \tag{6}$$

where V and A represent the volume and the surface of the piezoelectric continuum.

$\int_V \sigma_i \delta\varepsilon_i dV$ and $\int_V D_i \delta\phi_{,i} dV$ represent the virtual work done respectively by the internal forces and the electric field. $\int_A p_i \delta u_i dA$ and $\int_A q\delta\phi dA$ are the virtual external works done by the applied surface tractions p_i and charge density q respectively. σ_i, ε_i, u_i and D_i denote respectively the components of stress, strain, displacement and electric displacement.

The electromechanical equilibrium equation in terms of primary field variables is derived in Kapuria (2001) [8].

$$\left[L_{ij}\right]\bar{U} = P \tag{7}$$

E. Static Analysis

The boundary conditions at simply supported ends are:

$$N_x = w_0 = M_x = P_x = \phi^j = S_x^j = 0,$$
$$j = 1, \ldots, N \text{ at } x = 0, a. \tag{8}$$

The solution of the governing equation (7) which identically satisfies the boundary conditions as given by (8) is expressed in terms of Fourier Series as:

$$(w_0, \phi^j, N_x, M_x, P_x, S_x^j, G^j) =$$

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$$\sum_{n=1}^{\infty}(w_0,\phi^j,N_x,M_x,P_x,S_x^j,G^j)_n \sin \bar{n}x$$

$$(u_0,\psi_1,Q_x,\bar{Q}_x^j,H^j)=\sum_{n=1}^{\infty}(u_0,\psi_1,Q_x,\bar{Q}_x^j,H^j)_n \cos \bar{n}x$$

with $\bar{n} = n\pi / a$. Substituting these expressions into (7) yields the coupled system of linear algebraic expressions for the *n*th Fourier component which can be partitioned and arranged in the following form:

$$\begin{bmatrix} X^{uu} & X_{ss}^{ue} \\ X_{ss}^{eu} & X_{ss}^{ee} \end{bmatrix} \begin{pmatrix} U^n \\ \Phi^n \end{pmatrix} = \begin{pmatrix} F^n - X_{sa}^{ue}\Phi_a^n \\ Q_s^n - X_{sa}^{ee}\Phi_a^n \end{pmatrix} \tag{9}$$

III. Results

Results are presented for simply supported hybrid beams. The thickness of piezoelectric layer is varied from 0.05*h* to 0.30*h*, where *h* is the total thickness of the beam. The beam configuration is as shown in Fig. 2. Beam (a) is a symmetric 4-ply laminate of equal thicknesses of material 1 with orientation [0º/90º/90º/0º] while beam (b) is a sandwich beam with graphite-epoxy (material 2) faces and a soft core with thickness $8h_f$, where, h_f is thickness of face. Beam (c) is an anti-symmetric 4-ply laminate made up of material 3 with lay-up [90º/0º/90º/0º]. The total thickness of the beam is constant. The material properties are given in Table 1.

The results are plotted for uniformly distributed load as well as sinusoidal loading conditions in Figs. 3 through 5.

Two load cases are considered:

1. Uniform pressure $p_z^2 = -p_0$ on top.
2. Uniform applied potential $\phi^{n_e} = \phi_0$ on the top.

Similarly for sinusoidal loading conditions:

1. A sinusoidal pressure on top surface, $p_z^2 = p_0 \sin(\pi x / a)$.
2. An actuating potential applied at the top surface, $\phi^N = \phi_0 \sin(\pi x / a)$.

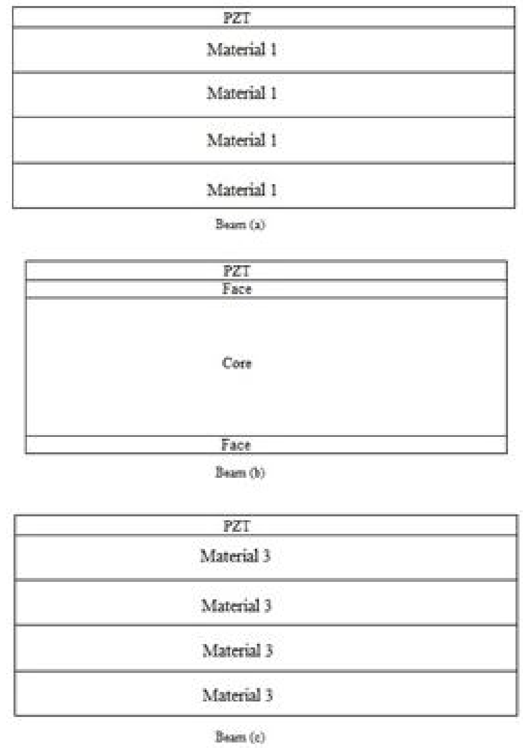

Figure 2. Beam Configurations

The results are non-dimensionalised with $S = a/h$, $d_T = 374 \times 10^{-12}$ CN^{-1}, $Y_T = 10.3$ GPa for beam (a) and (c) and 6.9 GPa for beam (b):

1. $\bar{w} = 100 w Y_T / h S^4 p_0$, $\bar{\sigma}_x = \sigma_x / S^2 p_0$.
2. $\bar{w} = 10 w / S^2 d_T \phi_0$, $\bar{\sigma}_x = \sigma_x h / 10 Y_T d_T \phi_0$, $\bar{D}_z = D_z h / 100 Y_T d_T^2 \phi_0$.

Table I. Material Properties

Property	Units	Material 1	Material 2	Material 3	Core	PZT-5A
Y_1	GPa	181	172.5	181	0.276	61
Y_2	GPa	10.3	6.9	10.3	0.276	61
Y_3	GPa	10.3	6.9	10.3	3.45	53.2
G_{12}	GPa	7.17	3.45	7.17	0.1104	22.6
G_{23}	GPa	2.87	1.38	2.87	0.414	21.1
G_{31}	GPa	7.17	3.45	7.17	0.414	21.1
ν_{12}		0.25	0.25	0.28	0.25	0.35
ν_{13}		0.25	0.25	0.28	0.02	0.38
ν_{23}		0.33	0.25	0.33	0.02	0.38
$d_{31} = d_{32}$	pm/V					- 171
d_{33}	pm/V					374
$d_{15} = d_{24}$	pm/V					584
$\eta_{11} = \eta_{22}$	nF/m					15.3
η_{33}	nF/m					15.0

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The effect of the beam lay-up is also studied by obtaining the results for symmetric [0º/45º/45º/0º] as well as anti-symmetric angle ply beams [45º/-45º/45º/-45º]. The results are also obtained for a beam with piezo-layer attached to both top and bottom surfaces. These results are plotted in Figs. 6 through 9.

Conclusions

An efficient coupled electromechanical model has been presented for static analysis of hybrid beams with embedded or surface bonded piezoelectric layers under general electromechanical loading. This model combines a third order in-plane displacement and a layerwise linear electric field such that continuity of transverse shear stress at the interface of the layers is preserved with shear free conditions at top and bottom surfaces. The no. of primary variables is same as in FSDT and hence the computational cost is independent of no. of layers in the laminate.

With increase in thickness of piezoelectric layer, the deflection of the beam increases and the normal stress at the top surface decreases as shown in the figures. It also reveals that the piezoelectric coupling effect increases with increase in piezo-layer thickness to overall thickness ratio. It is also observed that the deflection for anti-symmetric angle-ply laminates is more than that of cross ply and symmetric angle-ply laminates. The stresses increase for load case 1 while they decrease for load case 2. For beams with piezo-layers on both surfaces, deflection increases as the thickness of the piezo-layers is increased.

Overall, the presented theory is well suited to model sensory as well as active response of smart composite beams for all applications.

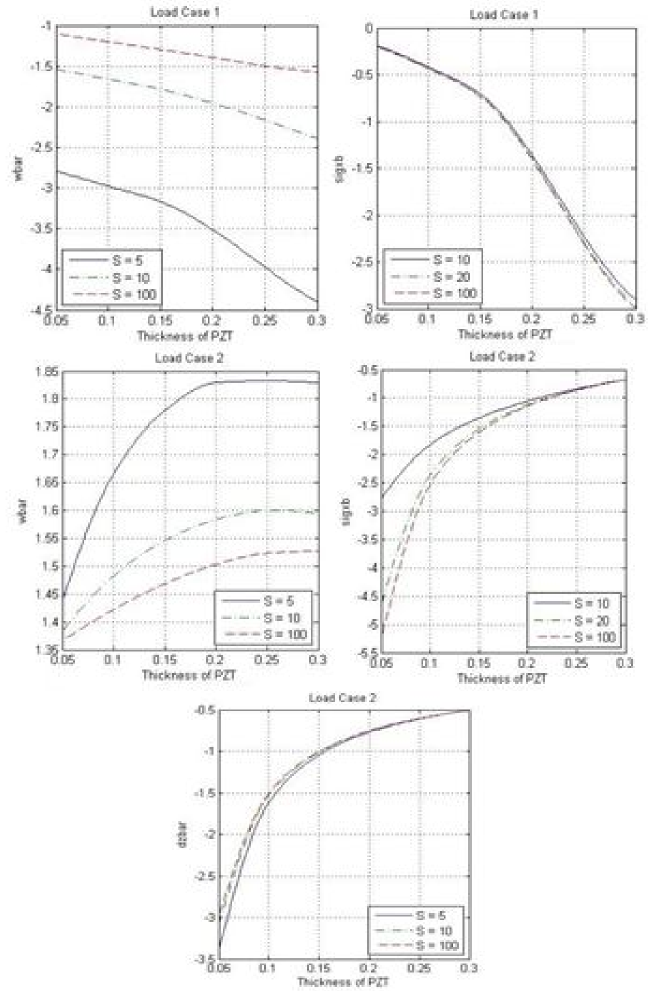

Figure 3. Variation of $\overline{w}$, $\overline{\sigma}$ and $\overline{D}_z$ in beam (a) with thickness of piezo-layer for uniformly distributed loading

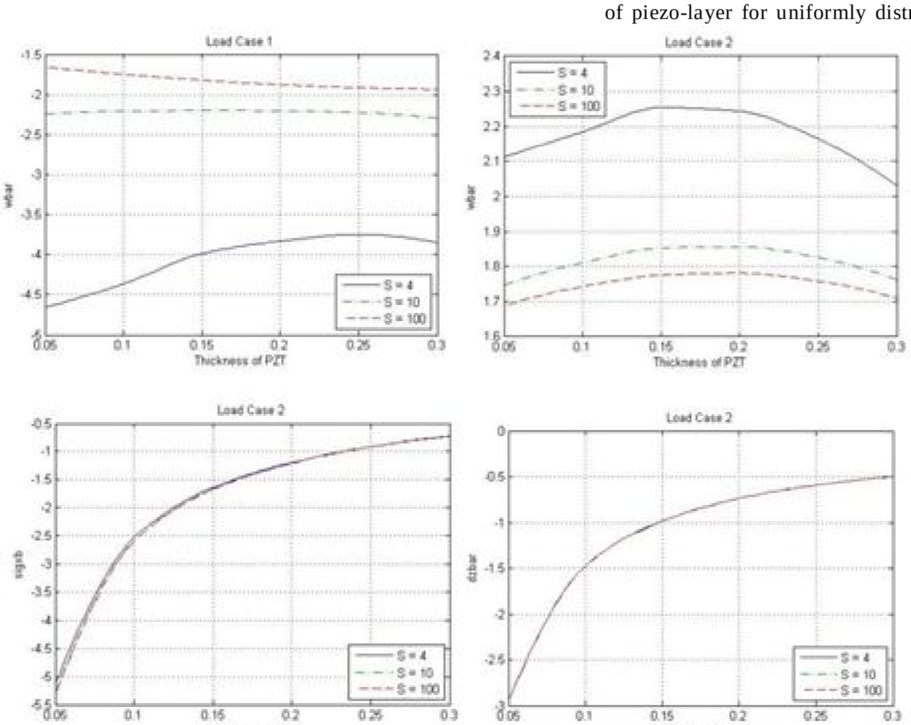

Figure 4. Variation of $\overline{w}$, $\overline{\sigma}$ and $\overline{D}_z$ in beam (c) with thickness of piezo-layer for sinusoidal loading

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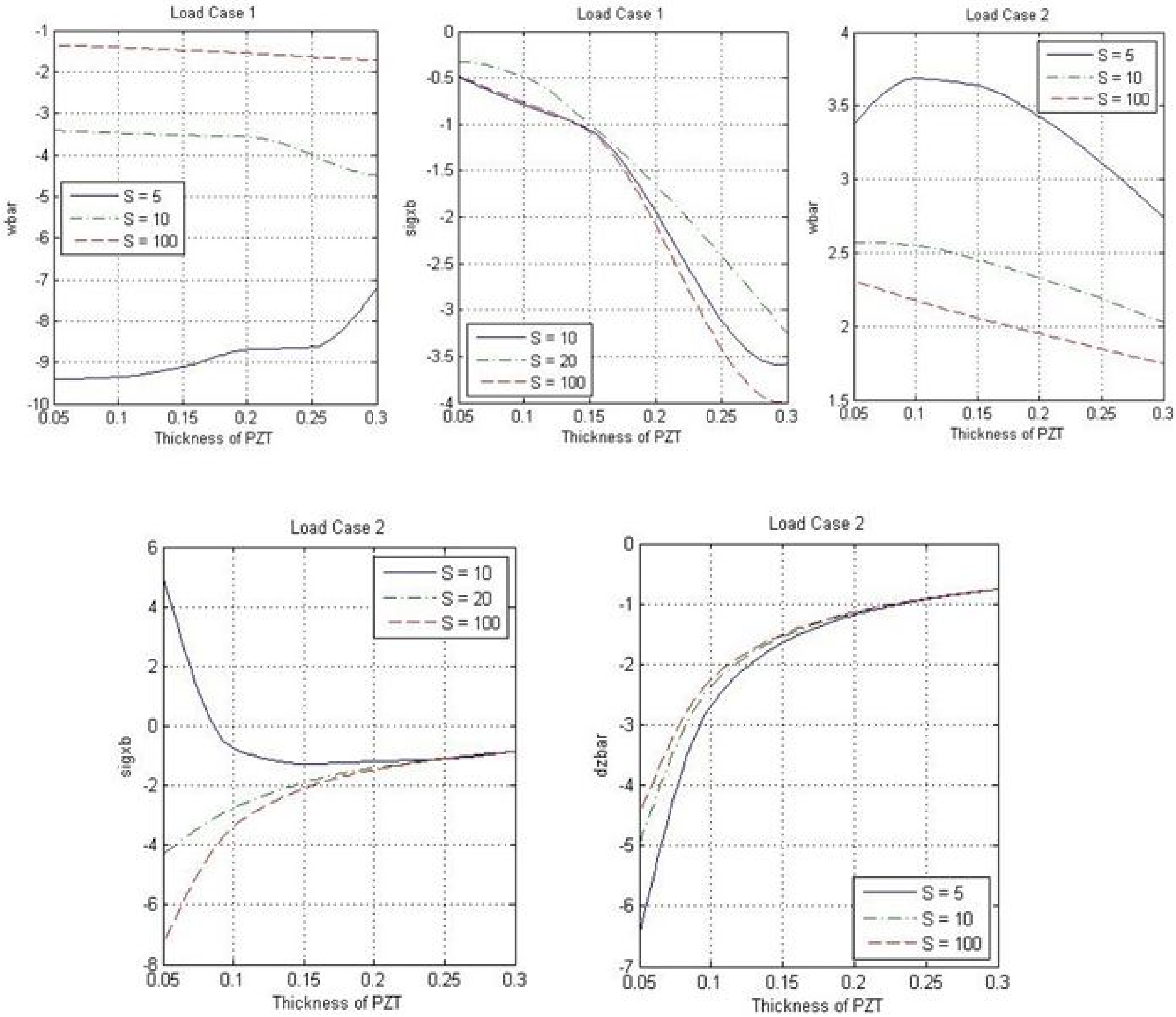

Figure 5. Variation of $\overline{w}$, $\overline{\sigma}$ and $\overline{D}_z$ in beam (b) with thickness of piezo-layer for uniformly distributed loading

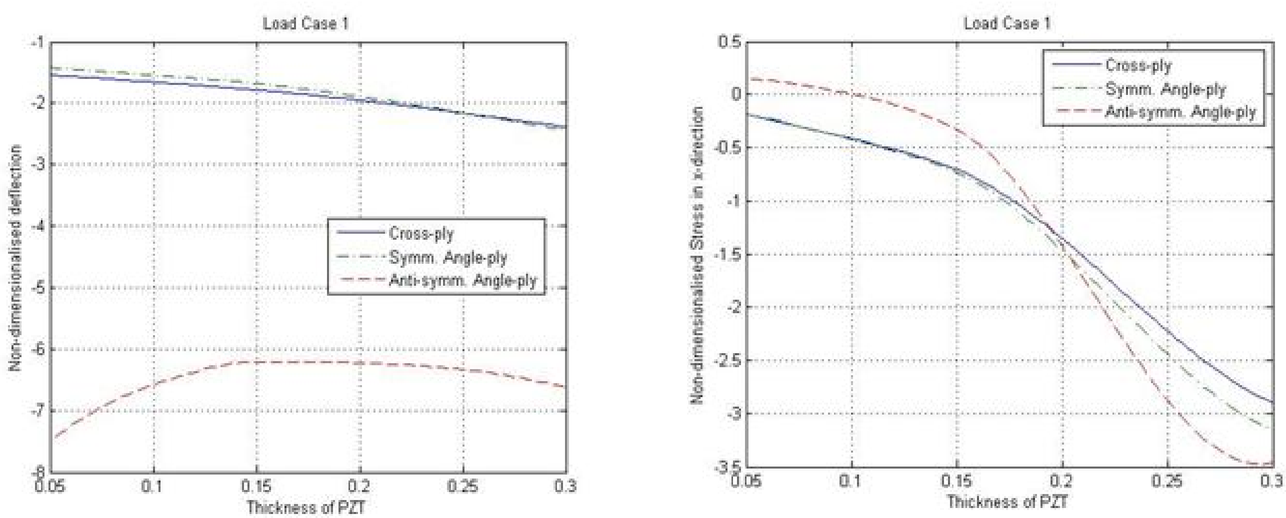

Figure 6. Variation of $\overline{w}$ and $\overline{\sigma}$ in beam (a) with thickness of piezo-layer for uniformly distributed load case 1

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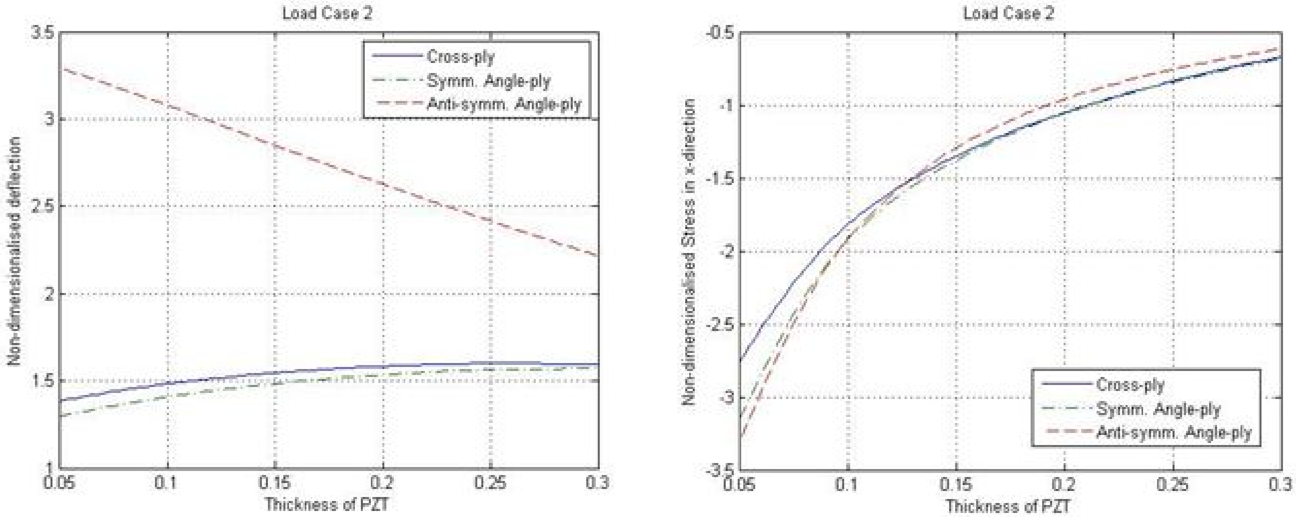

Figure 7. Variation of $\overline{w}$ and $\overline{\sigma}$ in beam (a) with thickness of piezo-layer for uniformly distributed load case 2

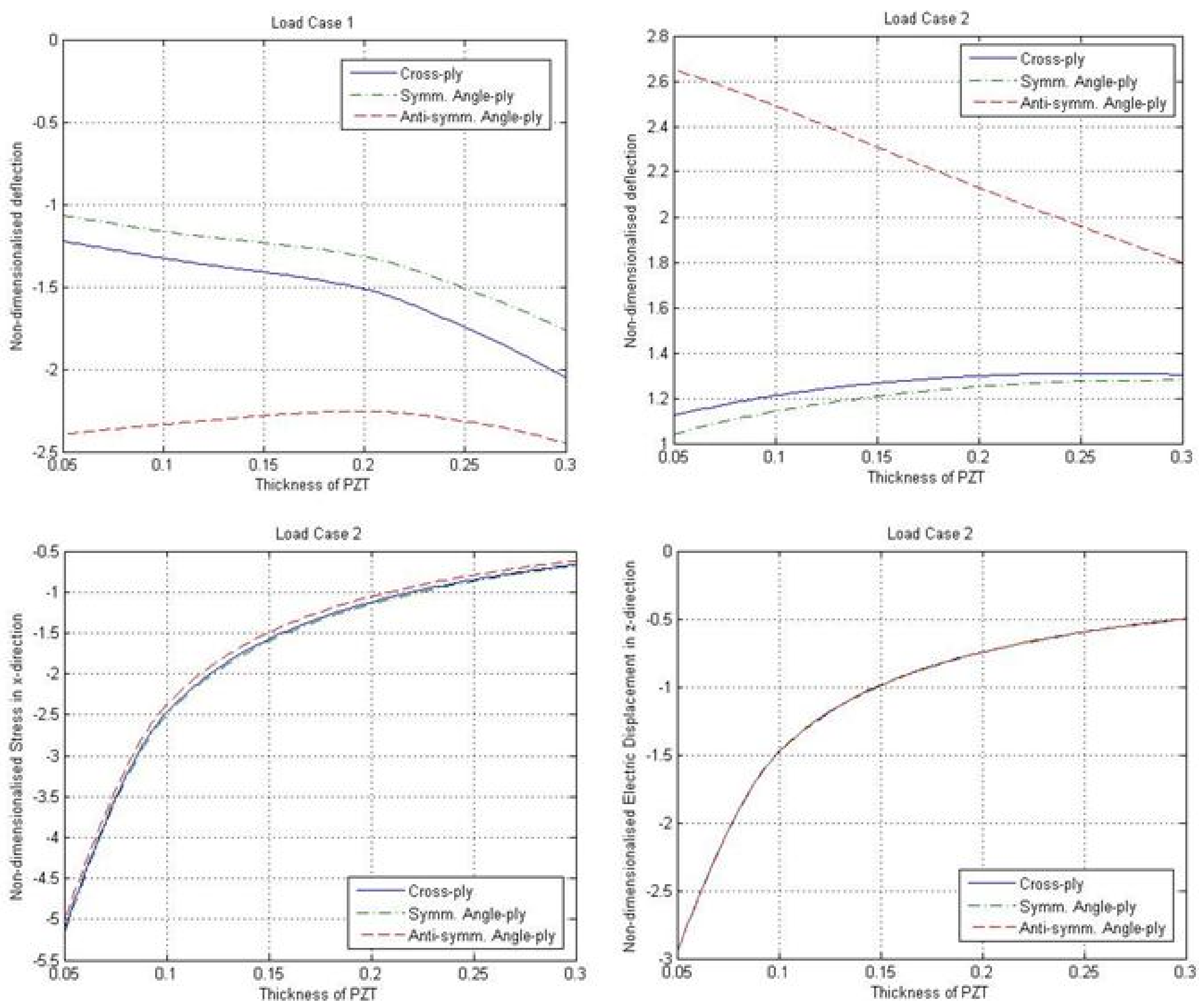

Figure 8. Variation of $\overline{w}$, $\overline{\sigma}$ and $\overline{D}_z$ in beam (a) with thickness of piezo-layer for sinusoidal loading

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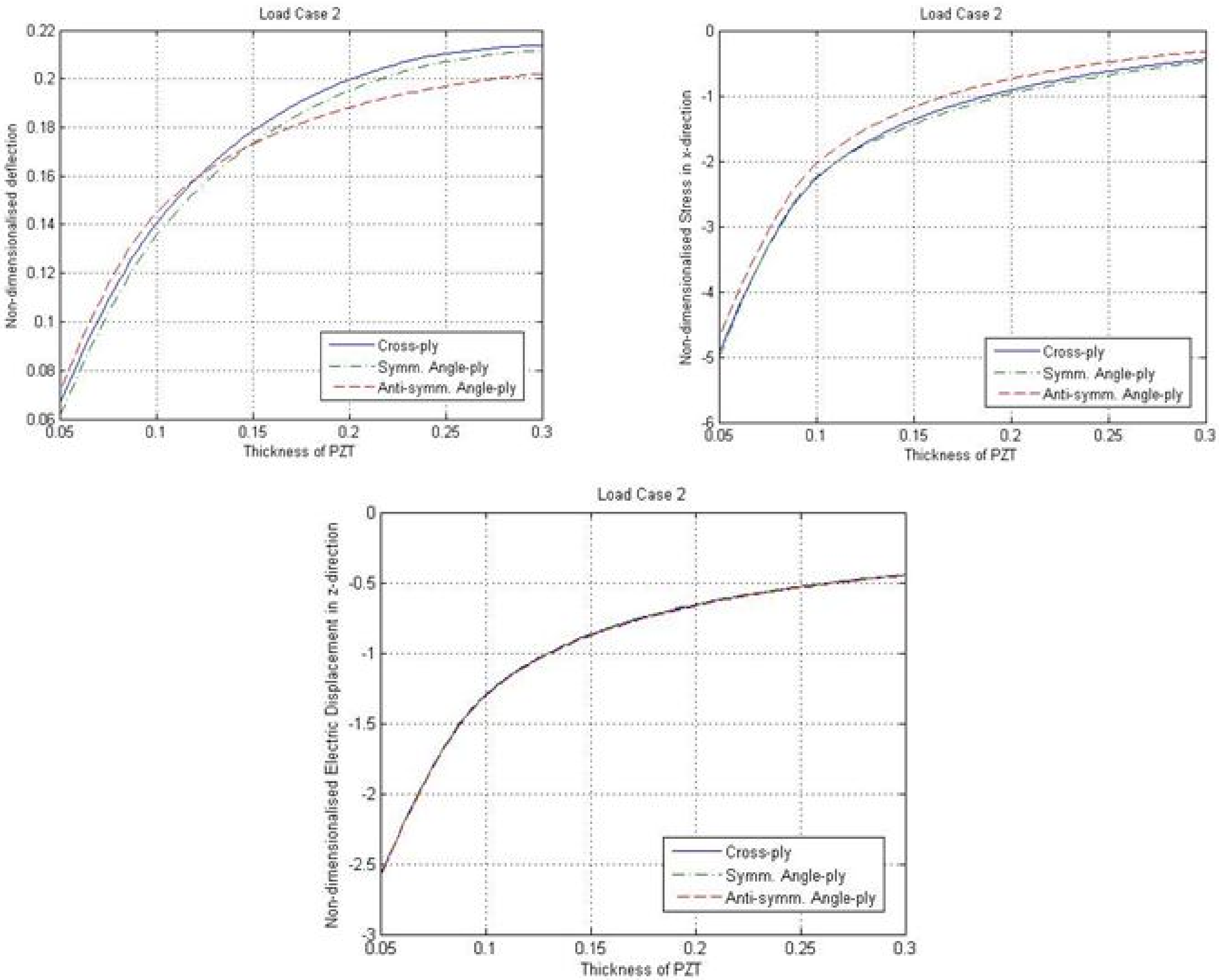

Figure 9. Variation of $\overline{w}$, $\overline{\sigma}$ and $\overline{D}_z$ in beam with thickness of piezo-layer on both surfaces for sinusoidal loading

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